MATH 2050 - Limits of sequences

(Reference: Bartle § 3.1)

GOAL: Define limit of a sequence $\lim (x_n) = x$ and study limit properties

Def$^{\underline{n}}$: A sequence of real numbers is a function

$$X : \mathbb{N} \longrightarrow \mathbb{R}$$

Denote: $X(1) := x_1$, $X(2) =: x_2$, ..., $X(n) =: x_n$

Write: $X = (x_n) = (x_1, x_2, x_3, \cdots)$

CAUTION: Sets $\neq$ Sequences

E.g.) $((-1)^n) = (-1, 1, -1, 1, \dots)$ ordered & infinite

$\{(-1)^n : n \in \mathbb{N}\} = \{-1, 1\}$ unordered & maybe finite

Examples of sequences

(1) constant seq. $(1, 1, 1, 1, 1, \dots)$

(2) geometric seq. $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right) = \left(\frac{1}{2^n}\right)$

(3) arithmetic seq. $(1, 5, 9, 13, 17, \dots) = (4n-3)$

→ +ve odd seq. $(1, 3, 5, 7, 9, \dots)$

+ve even seq. $(2, 4, 6, 8, 10, \dots)$

(4) Fibonacci seq. ("inductively defined")

$$x_1 := 1 \quad ; \quad x_2 := 1$$

$$x_n := x_{n-1} + x_{n-2} \quad \text{for } n \geqslant 3$$

$$(x_n) = (1, 1, 2, 3, 5, 8, 13, 21, \dots)$$

Q: How to define the limit of a sequence?

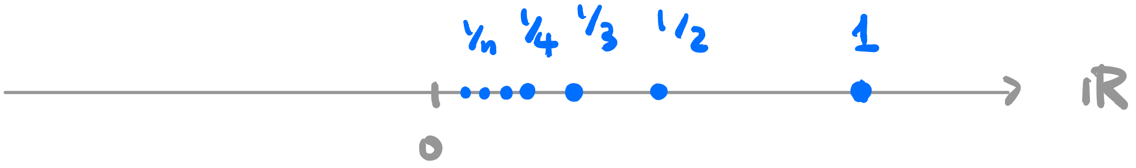
Simplest example:

$$(x_n) = \left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{100}, \dots\right)$$

We want to say: "$\lim \left(\frac{1}{n}\right) = 0$"

because x_n "eventually" are getting "close" to 0

quantify this!



Def$^{\underline{n}}$: (ε - K definition for limit)

We say that (x_n) converges to $x \in \underline{\mathbb{R}}$ (finite number)

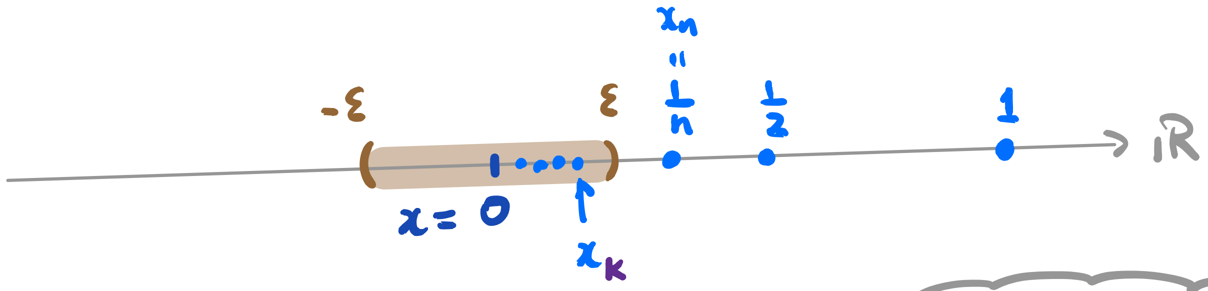
Notation: $\lim (x_n) = x$ or $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$

iff $\forall \overset{(\text{small})}{\varepsilon} > 0$, $\exists \overset{(\text{large})}{K} = K(\varepsilon) \in \mathbb{N}$ s.t.

$$\underline{|x_n - x| < \varepsilon} \qquad \forall\, n \geq K$$

(i.e. $x - \varepsilon < x_n < x + \varepsilon$)

Picture:



Example: $\boxed{\lim(\frac{1}{n}) = 0}$

$|\frac{1}{n} - 0| < \varepsilon$

$\iff \frac{1}{n} < \varepsilon$

$\iff \frac{1}{\varepsilon} < n$

Let $\varepsilon > 0$ be fixed but arbitrary.

By Archimedean Property, we can choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{\varepsilon}$. Then, $\forall\, n \geq K$,

$$|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{K} < \varepsilon$$

$\square$

Example: $$\boxed{\lim\left(\frac{3n+2}{n+1}\right) = 3}$$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K \overset{*}{>} \frac{1}{\varepsilon}$ (by Archimedean Property)

Then, $\forall\, n \geq K$,

$$\left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{(3n+2) - 3(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right|$$

$$= \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{K} \overset{*}{<} \varepsilon$$

$\blacksquare$

Def$^{\underline{n}}$: Given a seq (x_n) of real numbers, we say

(i) (x_n) is convergent if $\exists\, x \in \mathbb{R}$ s.t. $\lim(x_n) = x$

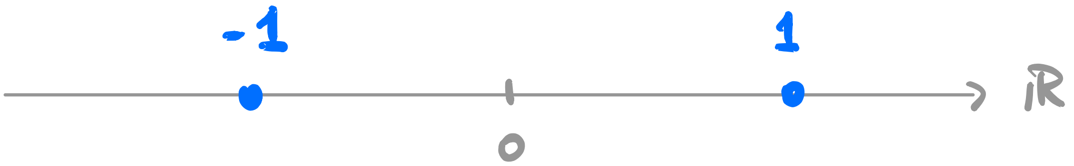
(ii) (x_n) is divergent if (x_n) is NOT convergent

i.e. $\cancel{\exists}\, x \in \mathbb{R}$ s.t. $\lim(x_n) = x$

Example of divergent seq

Example: $((-1)^n)$ is divergent.

Picture:



Why 1 is not the limit?

If $\lim((-1)^n) = 1$, then

$$\left[\forall \varepsilon > 0,\ \exists K \in \mathbb{N} \text{ st } |(-1)^n - 1| < \varepsilon \overset{\forall n \geq K}{\curlyvee} \right] \cdots (*)$$

But for n odd, $|(-1)^n - 1| = |(-1) - 1| = 2$

So $(*)$ cannot be true.

Take $\varepsilon = 1 > 0$, if $(*)$ is true, $\exists K = K(1) \in \mathbb{N}$.

but if we choose $n \geq K$ odd, then

$$|(-1)^n - 1| = 2 \not< 1$$ Contradiction!

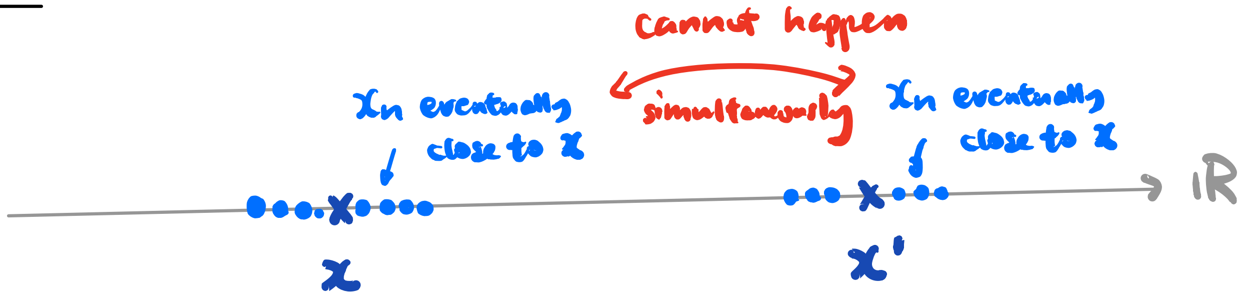
Similarly, -1 is NOT the limit either (Pf: Exercise)

Actually, $\nexists\, x \in \mathbb{R}$ st $\lim((-1)^n) = x$.

(Pf later).

Prop: Any convergent seq. (x_n) has a unique limit.

Proof: Idea: Suppose there are two limits x, x'



Suppose NOT, then $\exists\ x \neq x' \in \mathbb{R}$ s.t.

$$\lim(x_n) = x \quad \& \quad \lim(x_n) = x'$$

Consider $\varepsilon := \dfrac{|x-x'|}{4} > 0$. by def^n of limit,

- $\lim(x_n) = x \Rightarrow \exists\ K = K(\varepsilon) \in \mathbb{N}$ s.t.

  $$|x_n - x| < \varepsilon \qquad \forall n \geq K$$

  for the same ε

- $\lim(x_n) = x' \Rightarrow \exists\ K' = K'(\varepsilon) \in \mathbb{N}$ s.t.

  $$|x_n - x'| < \varepsilon \qquad \forall n \geq K'$$

Take $\overline{K} := \max\{K, K'\} \in \mathbb{N}$. Then, $\forall n \geq \overline{K}$,

$$|x - x'| = |(x_n - x) - (x_n - x')| \qquad \text{Contradiction!}$$

$$\overset{(\triangle\text{-ineq})}{\leq} |x_n - x| + |x_n - x'| < \varepsilon + \varepsilon = \frac{|x-x'|}{2}$$

□

Now, we come back to

Claim: $((-1)^n)$ is divergent.

Pf: Suppose NOT, i.e. $((-1)^n)$ is convergent.

By Prop. $\exists$ unique limit $x = \lim((-1)^n)$.

By def$^{\underline{n}}$ of limit, $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t.

$$|(-1)^n - x| < \varepsilon \qquad \forall n \geq K.$$

Fix $\varepsilon = 1 > 0$, then $\exists K \in \mathbb{N}$ s.t.

$$|(-1)^n - x| < 1 \qquad \forall n \geq K.$$

Fix $n_1, n_2 \geq K$ s.t. n_1 odd, n_2 even. Then

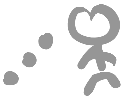
$$\begin{aligned} 2 = |-1-1| &= |(-1)^{n_1} - (-1)^{n_2}| \\ &= |((-1)^{n_1} - x) - ((-1)^{n_2} - x)| \\ &\leq |(-1)^{n_1} - x| + |(-1)^{n_2} - x| \\ &< 1 + 1 = 2 \end{aligned}$$

Contradiction! $\square$

More examples

Example: Let $a > 0$. Then $\boxed{\lim\left(\frac{1}{1+na}\right) = 0}$

Let $\varepsilon > 0$ be fixed but arbitrary.



Choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{a\varepsilon} > 0$

$$\left|\frac{1}{1+na} - 0\right| < \varepsilon$$

$$\parallel$$

$$\frac{1}{1+na} < \frac{1}{na} \leq \underline{\underline{\frac{1}{Ka} < \varepsilon}}$$

$$\Downarrow$$

$$K > \frac{1}{a\varepsilon}$$

Then, $\forall n \geq K$.

$$\left|\frac{1}{1+na} - 0\right| = \left|\frac{1}{1+na}\right|$$

$$= \frac{1}{1+na} < \frac{1}{na} \leq \frac{1}{Ka} < \varepsilon$$

□

Example: Let $b \in (0,1)$. Then $\boxed{\lim(b^n) = 0}$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{\log \varepsilon}{\log b}$

Want: $|b^n - 0| < \varepsilon$

$$|b^n - 0| = b^n \leq \underline{\underline{b^K < \varepsilon}}$$

Solve for K

$$K \log b < \log \varepsilon$$

$$K > \frac{\log \varepsilon}{\log b}$$

Then, $\forall n \geq K$.

$$|b^n - 0| = b^n \leq b^K < \varepsilon$$

□

Recall: $\lim(x_n) = x$ iff

$$\forall \varepsilon > 0, \ \exists K \in \mathbb{N} \text{ s.t. } |x_n - x| < \varepsilon \quad \forall n \geq K$$

Q: Given $\underset{\substack{\uparrow \\ \text{small}}}{\varepsilon > 0}$, how to find such a $\underset{\substack{\uparrow \\ \text{large}}}{K}$?

Some useful tools (inequalities)

(i) "Fraction comparison" (everything > 0)

$$\frac{\text{smaller}}{\text{Bigger}} \leq \frac{\square}{\square} \leq \frac{\text{Bigger}}{\text{smaller}}$$

(ii) Triangle ineq. / Reverse Triangle ineq. / AM-GM ineq.

(iii) Bernoulli's ineq: $(1+x)^n \geq 1+nx$ $\quad \forall x \geq -1$, $\forall n \in \mathbb{N}$

Example 1 (revisted)

Let $b \in (0,1)$ be fixed. Then $\boxed{\lim(b^n) = 0}$

Pf: Since $b \in (0,1)$, we can write it in the form

$$b = \frac{1}{1+a} \quad \text{for some } a > 0$$

$(\because b < 1)$

Raise to n^{th} power, apply Bernoulli's ineq.

$$b^n = \left(\frac{1}{1+a}\right)^n = \frac{1}{(1+a)^n} \overset{\downarrow}{\leq} \frac{1}{1+na}$$

By the example on Tuesday.

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{a\varepsilon}$. Then $\forall n \geq K$

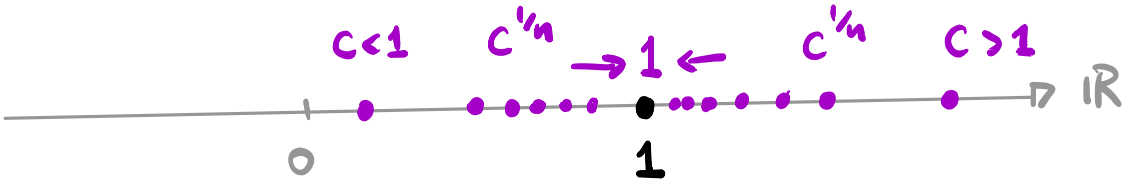
$$|b^n - 0| \leq \frac{1}{1+na} \leq \frac{1}{1+Ka} < \frac{1}{Ka} < \varepsilon$$

$\square$

Example 2: Let $c > 0$ be fixed. Then,

$$\boxed{\lim (c^{\frac{1}{n}}) = 1}$$

Picture:



Proof: Case 1: $c = 1$ then $(c^{\frac{1}{n}}) = (1)$ const seq.

Case 2: $c > 1$

Recall: $c^{\frac{1}{n}} > 1 \;\; \forall n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$,

$$c^{\frac{1}{n}} = 1 + d_n \quad \text{for some } \underline{d_n > 0}$$

Raise to n^{th} power, apply Bernoulli.

$$c = \left(c^{\frac{1}{n}}\right)^n = (1 + d_n)^n \overset{\because d_n > 0}{\geq} 1 + n d_n$$

Rearrange. $$d_n \leq \frac{c-1}{n} \qquad \text{————} \quad (*)$$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ st. $K > \frac{c-1}{\varepsilon}$.

When $n \geq K$, we have

$$\left| c^{\frac{1}{n}} - 1 \right| = |d_n| = d_n \leq \frac{c-1}{n} \leq \frac{c-1}{K} < \varepsilon$$

Case 3: $0 < c < 1$

Recall: $0 < c^{\frac{1}{n}} < 1 \quad \forall n \in \mathbb{N}.$

Write: $$c^{\frac{1}{n}} = \frac{1}{1 + h_n} \qquad \text{where } h_n > 0$$

Raise to n^{th} power.

$$c = \left(c^{\frac{1}{n}}\right)^n = \frac{1}{(1+h_n)^n} \leq \frac{1}{1 + n h_n} < \frac{1}{n h_n}$$

Rearrange. $$h_n < \frac{1}{cn} \qquad \text{for all } n \in \mathbb{N}$$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{c\varepsilon}$.

For $n \geq K$,

$$|c^{\frac{1}{n}} - 1| = \left| \frac{1}{1+h_n} - 1 \right| = \left| \frac{-h_n}{1+h_n} \right|$$

$$= \frac{h_n}{1+h_n} < h_n < \frac{1}{cn} \leq \frac{1}{cK} < \varepsilon$$

$\square$

Example 3: $\boxed{\lim (n^{\frac{1}{n}}) = 1}$

Pf: Recall: $1 \leq n^{\frac{1}{n}}$ $\forall n \in \mathbb{N}$. Write $n^{\frac{1}{n}} = 1 + \overset{\geq 0}{\boxed{k_n}}$

$$\Rightarrow \quad n = (1+k_n)^n \underset{(\because k_n \geq 0)}{\geq} 1 + \frac{1}{2} n(n-1) k_n^2$$

Rearrange $k_n^2 \leq \frac{n-1}{\frac{1}{2}n(n-1)} = \frac{2}{n}$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{2}{\varepsilon^2}$. Then $\forall n \geq K$,

$$|n^{\frac{1}{n}} - 1| = |k_n| = k_n \leq \sqrt{\frac{2}{n}} \leq \sqrt{\frac{2}{K}} < \varepsilon$$

$\square$